1. (10 total points) Evaluate the following indefinite integrals.

(a) (5 points) $\int t^5 \sin(t^3)\, dt$

$x = t^3$

$dx = 3t^2 dt$

$\frac{1}{3t^2} dx = dt$

$$\frac{1}{3} \int t^{\overset{3}{\cancel{5}}} \sin(x) \frac{1}{3\cancel{t^2}} dx$$

$$\frac{1}{3} \int x \sin(x)\, dx$$

$u = x$ $\quad dv = \sin(x)dx$

$du = dx$ $\quad v = -\cos(x)$

$$\frac{1}{3}\left[-x\cos(x) - \int -\cos(x)\,dx\right]$$

$$-\frac{1}{3}x\cos(x) + \frac{1}{3}\sin(x) + C$$

$$\boxed{-\frac{1}{3}t^3\cos(t^3) + \frac{1}{3}\sin(t^3) + C}$$

(b) (5 points) $\int \frac{1}{x(x+\sqrt{x})}\, dx$

$t^2 = x$

$2t\,dt = dx$

$$\int \frac{1}{t^{\cancel{2}}(t^2+t)} 2\cancel{t}\,dt$$

$$\frac{2}{t(t^2+t)} = \frac{2}{t^2(t+1)} = \frac{A}{t} + \frac{B}{t^2} + \frac{C}{t+1}$$

$\Rightarrow \quad 2 = At(t+1) + B(t+1) + Ct^2$

$t=0 \Rightarrow B = 2$

$t=-1 \Rightarrow C = 2$

AND $A + C = 0 \Rightarrow A = -C = -2$

$$\int \frac{-2}{t} + \frac{2}{t^2} + \frac{2}{t+1}\, dt$$

$$-2\ln|t| - \frac{2}{t} + 2\ln|t+1| + C$$

$$= \boxed{-2\ln\sqrt{x} - \frac{2}{\sqrt{x}} + 2\ln(|\sqrt{x}|+1) + C}$$

2. (10 total points) Evaluate the following definite integrals.

(a) (5 points) $\int_0^1 \ln(1+t^2)\,dt$

$u = \ln(1+t^2) \qquad dv = dt$

$du = \frac{2t}{1+t^2}dt \qquad v = t$

$$t\ln(1+t^2)\Big|_0^1 - \int_0^1 \frac{2t^2}{1+t^2}dt$$

$$t^2+1 \overline{\smash{\big)}\,2t^2} \quad 2 \qquad -(2t^2+2) \qquad -2$$

$$(\ln(2) - 0) - \int_0^1 2 - \frac{2}{1+t^2}\,dt$$

$$\ln(2) - \left[2t - 2\tan^{-1}(t)\Big|_0^1\right]$$

$$\ln(2) - [(2 - 2\tan^{-1}(1)) - (0-0)]$$

$$\ln(2) - 2 + 2\pi/4 = \boxed{\ln(2) - 2 + \pi/2}$$

(b) (5 points) $\int_0^2 \frac{x^3}{\sqrt{4+x^2}}\,dx$

$u = 4 + x^2 \qquad x^2 = 4 - u$

$du = 2x\,dx$

$$\int_4^8 \frac{\overset{2}{\cancel{x^3}}}{\sqrt{u}}\,\frac{1}{2\cancel{x}}\,du$$

$$\frac{1}{2}\int_4^8 \frac{(4-u)}{\sqrt{u}}\,du = \frac{1}{2}\int_4^8 4u^{-1/2} - u^{1/2}\,du$$

$$= \frac{1}{2}\left[8u^{1/2} - \frac{2}{3}u^{3/2}\Big|_4^8\right]$$

$$= \left(4(8)^{1/2} - \frac{1}{3}(8)^{3/2}\right) - \left(4\cdot 2 - \frac{1}{3}(4)^{3/2}\right)$$

$$= 4\sqrt{8} - \frac{1}{3}8^{3/2} - (8 - 8/3)$$

$$= 4\sqrt{8} - \frac{8}{3}\sqrt{8} - \frac{16}{3} = \frac{4}{3}\sqrt{8} - \frac{16}{3}$$

$$= \boxed{\frac{4}{3}(\sqrt{8} - 4)} = \frac{4}{3}(2\sqrt{2} - 4) = \frac{8}{3}(\sqrt{2} - 2)$$

1. Evaluate the following indefinite integrals.

(a) (5 points) $\int \sin(x)\sqrt{\cos(x)}\,dx$

$u = \cos(x)$

$du = -\sin(x)\,dx$

$= -\int \sqrt{u}\,du$

$= -\frac{2}{3}u^{3/2} + C$

$= \boxed{-\frac{2}{3}\cos^{3/2}(x) + C}$

(b) (5 points) $\int \sqrt{3-2x-x^2}\,dx$

$3-2x-x^2$

$= 3-(2x+x^2)$

$= 3-(x^2+2x+1-1)$

$= 3-((x+1)^2-1)$

$= 3-(x+1)^2+1 = 4-(x+1)^2$

$= \int \sqrt{4-(x+1)^2}\,dx$

$\downarrow \qquad \downarrow$

$= \int 2\cos\theta \cdot 2\cos\theta\,d\theta$

$= 4\int \frac{1}{2}(1+\cos(2\theta))\,d\theta$

$= 2\left[\theta + \frac{1}{2}\sin(2\theta)\right] + C$

$= 2\left[\theta + \sin\theta\cos\theta\right] + C$

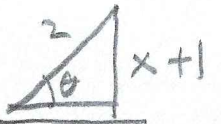
$= 2\theta + 2\sin\theta\cos\theta + C$

$= 2\sin^{-1}\left(\frac{x+1}{2}\right) + 2\,\frac{x+1}{2}\,\frac{\sqrt{3-2x-x^2}}{2} + C$

$= \boxed{2\sin^{-1}\left(\frac{x+1}{2}\right) + \frac{1}{2}(x+1)\sqrt{3-2x-x^2} + C}$

$x+1 = 2\sin\theta$

$dx = 2\cos\theta\,d\theta$



$\sqrt{4-(x+1)^2} = \sqrt{3-2x-x^2}$

2. Evaluate the following definite integrals.

(a) (5 points) $\int_0^{\pi} \sec\left(\frac{x}{3}\right) \cdot \tan^3\left(\frac{x}{3}\right)\, dx$

$t = \frac{1}{3}x$

$dt = \frac{1}{3}dx$

$3\int_0^{\pi/3} \sec(t)\tan^3(t)\,dt$

$3\int_0^{\pi/3} \tan^2(t)\sec(t)\tan(t)\,dt$

$\sec(0) = 1$

$\sec(\pi/3) = 2$

$3\int_0^{\pi/3} (\sec^2(t) - 1)\sec(t)\tan(t)\,dt$

$u = \sec(t)$

$du = \sec(t)\tan(t)\,dt$

$3\int_1^2 (u^2-1)\,du$

$3\left(\frac{1}{3}u^3 - u\Big|_1^2\right) = u^3 - 3u\Big|_1^2$

$= (8-6) - (1-3)$

$= 2 - (-2) = \boxed{4}$

(b) (5 points) $\int_{-1}^{2} \frac{x}{x^2+2x+10}\, dx$

$x^2 + 2x + 1 - 1 + 10$

$(x+1)^2 + 9$

$\int_{-1}^{2} \frac{x}{(x+1)^2+9}\,dx$

$t = x+1$

$dt = dx$

$\int_0^3 \frac{t-1}{t^2+9}\,dt$

$\int_0^3 \frac{t}{t^2+9}\,dt - \int_0^3 \frac{1}{t^2+9}\,dt$

$u = t^2+9$

$du = 2t\,dt$

$\frac{1}{2}\int_9^{18} \frac{1}{u}\,du - \frac{1}{3}\tan^{-1}(t/3)\Big|_0^3$

$\frac{1}{2}\ln|u|\Big|_9^{18} - \frac{1}{3}(\tan^{-1}(1) - \tan^{-1}(0))$

$\frac{1}{2}(\ln(18) - \ln(9)) - \frac{1}{3}\,\pi/4 = \boxed{\frac{1}{2}\ln(2) - \frac{\pi}{12}}$

7. Let $\mathscr{R}$ be the region bounded by the curve $y = \sqrt{x}$, the line $x = 4$, and the x-axis.

   (a) (6 points) The region $\mathscr{R}$ is rotated around the line $x = 6$ to form a solid. Set up an integral for the volume of this solid using cylindrical shells and EVALUATE THE INTEGRAL.

$$\int_0^4 2\pi(6-x)\sqrt{x}\,dx$$

$$= 2\pi\int_0^4 6x^{1/2} - x^{3/2}\,dx$$

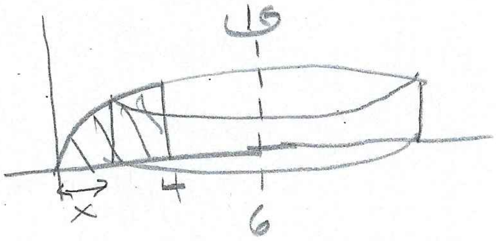
$$= 2\pi\left[6\,\frac{2}{3}x^{3/2} - \frac{2}{5}x^{5/2}\Big|_0^4\right]$$

$$= 2\pi\left[4(4)^{3/2} - \frac{2}{5}(4)^{5/2}\right]$$

$$= 2\pi\left[4\cdot 8 - \frac{2}{5}\,32\right]$$

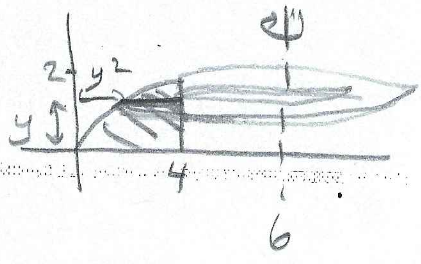
$$= 64\pi\left[1 - \frac{2}{5}\right]$$

$$= 64\pi\,\frac{3}{5} = \boxed{\frac{192}{5}\pi}$$



   (b) (4 points) Set up an integral for the volume of this solid using washers. DO NOT EVALUATE THE INTEGRAL.

$$\boxed{\int_0^2 \pi(6-y^2)^2 - \pi(2)^2\,dy}$$



3. (10 total points) A balloon is moving vertically up and down along a straight line above the ground, with the positive direction pointing up. The acceleration of the balloon at time t (in seconds) is given by $a(t) = -(t+5)$ ft/sec^2. The initial velocity of the balloon at time $t = 0$ is $v(0) = 12$ ft/sec.

(a) (3 points) Find the velocity $v(t)$ of the balloon as a function of time t.

$$v(t) = \int a(t)\,dt = \int -t - 5\,dt = -\tfrac{1}{2}t^2 - 5t + C$$

$$v(0) = 12 \Rightarrow C = 12$$

$$\boxed{v(t) = -\tfrac{1}{2}t^2 - 5t + 12} \quad \text{ft/sec}$$

(b) (4 points) Find the *total distance* traveled by the balloon from time $t = 0$ sec to time $t = 3$ sec.

$$\int_0^3 |v(t)|\,dt$$

$$v(t) \overset{?}{=} 0 \Rightarrow -\tfrac{1}{2}t^2 - 5t + 12 \overset{?}{=} 0$$

$$t^2 + 10t - 24 = 0$$

$$(t+12)(t-2) = 0$$

$$t = 2 \text{ or } t = -12$$

$$\int_0^2 -\tfrac{1}{2}t^2 - 5t + 12\,dt = -\tfrac{1}{6}t^3 - \tfrac{5}{2}t^2 + 12t\Big|_0^2$$

$$= -\tfrac{8}{6} - \tfrac{5}{2}(4) + 24 = -\tfrac{4}{3} - 10 + 24 = -4/3 + 14$$

$$\int_2^3 -\tfrac{1}{2}t^2 - 5t + 12\,dt = -\tfrac{1}{6}t^3 - \tfrac{5}{2}t^2 + 12t\Big|_2^3$$

$$= (-\tfrac{1}{6}27 - \tfrac{5}{2}9 + 36) - (-4/3 + 14)$$

-27

$$= -\tfrac{9}{2} - \tfrac{45}{2} + 36 + 4/3 - 14 = -\tfrac{54}{2} + 22 + 4/3$$

$$= -5 + 4/3$$

$$\text{TOTAL DIST} = -(-5 + 4/3) + (-4/3 + 14)$$

$$= 5 - 8/3 + 14 = 19 - 8/3 = \frac{57-8}{3} = \boxed{\frac{49}{3}} \text{ ft}$$

(c) (3 points) The balloon hits the ground at time $t = 6$ sec. What was its initial height above the ground at time $t = 0$?

$$h(6) = 0$$

$$h(t) = -\tfrac{1}{6}t^3 - \tfrac{5}{2}t^2 + 12t + D$$

$$\rightarrow -\tfrac{1}{6}(6)^3 - \tfrac{5}{2}(6)^2 + 12(6) + D = 0$$

$$\Rightarrow -36 - 5(18) + 72 + D = 0$$

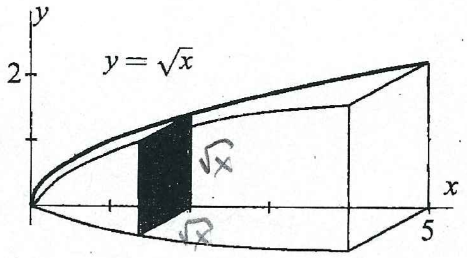
$$\Rightarrow -36 - 90 + 72 + D = 0$$

$$\Rightarrow -126 + 72 + D = 0$$

$$\Rightarrow -54 + D = 0$$

$$\boxed{D = 54 = h(0)} \quad \text{ft}$$

5. (4 points) Find the volume of the solid shown below. Each cross-section (slice) is a square.

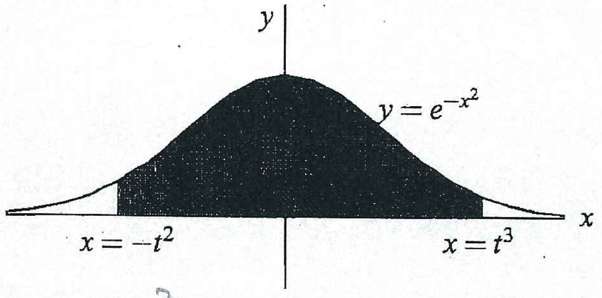


$$\int_0^5 \text{CROSS-SECTIONAL AREA}\, dx$$

$$= \int_0^5 (\sqrt{x})(\sqrt{x})\,dx = \int_0^5 x\,dx = \tfrac{1}{2}x^2\Big|_0^5$$

$$= \boxed{\frac{25}{2}}$$

6. (8 points) At each time $t \geq 0$, $\mathscr{R}_t$ is the region above the x-axis, below the curve $y = e^{-x^2}$, with left side on the line $x = -t^2$ and right side on the line $x = t^3$ (see the figure). Let $A(t)$ be the area of $\mathscr{R}_t$. Find $\frac{dA}{dt}$ at time $t = 1$.



$$A(t) = \int_{-t^2}^{t^3} e^{-x^2}\,dx$$

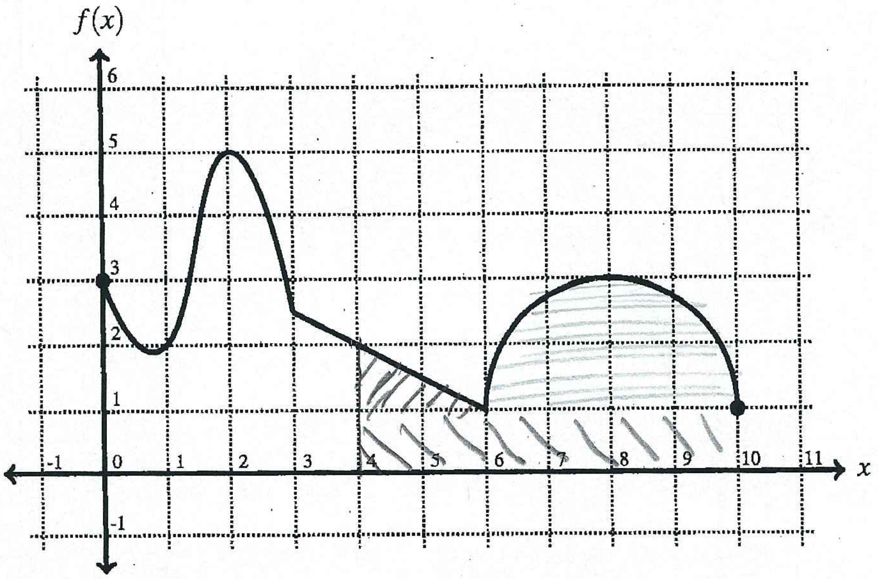
$$\frac{dA}{dt} = e^{-(t^3)^2}(3t^2) - e^{-(-t^2)^2}(-2t)$$

$$\frac{dA}{dt}\Big|_{t=1} = e^{-1}(3) - e^{-1}(-2)$$

$$= 5e^{-1}$$

$$= \boxed{5/e}$$

8. The graph of $f(x)$ is shown below. Use it to answer the following questions.



(a) (4 points) Compute the average value of $f(x)$ on the interval $[4, 10]$.

$$\text{AVERAGE} = \frac{1}{10-4}\underbrace{\int_4^{10} f(x)\,dx}_{\text{AREA UNDER CURVE FROM 4 to 10}}$$

$$\frac{1}{6}\left(\frac{1}{2}(1)(2) + (1)(6) + \frac{1}{2}\pi(2)^2\right) = \boxed{\frac{1}{6}(7+2\pi)}$$

(b) (6 points) Let $g(x) = \int_{x^2}^{7} f(t)\,dt$. Calculate $g''(2)$.

$$g'(x) = -f(x^2)2x = -2x f(x^2)$$

$$g''(x) = -2f(x^2) - 4x^2 f'(x^2)$$

$$\Rightarrow g''(2) = -2\underbrace{f(4)} - 16\underbrace{f'(4)}$$

$$= -2(2) - 16\left(-\frac{1}{2}\right)$$

$$= -4 + 8$$

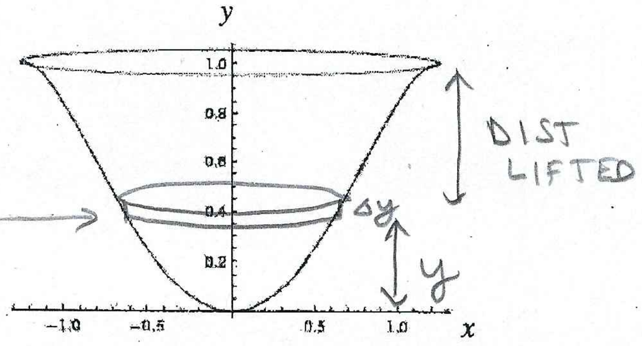
$$= \boxed{4}$$

$(4,2)$ $(6,1)$

$$\text{slope} = \frac{2-1}{4-6} = -\frac{1}{2}$$

6. (10 total points) The curve $x = \sqrt{\sin^{-1} y}$ for $0 \leq y \leq 1$ is rotated around the y-axis to form a container. The container is filled with a fluid that weighs 40 lb/ft^3. Length units for x and y are in feet.

VOLUME
OF SLICE
$= \pi(\text{RADIUS})^2 \Delta y$



(a) (6 points) Set up a definite integral (with respect to y) for the work (in ft-lb) required to empty the container by pumping all of the fluid to the top of the container.
(Note: Do not use the acceleration due to gravity; pounds are already a unit of force.)
IN THIS PART, DO NOT EVALUATE THE INTEGRAL YET.

$\text{FORCE} = 40\pi\left(\sqrt{\sin^{-1}(y)}\right)^2 \Delta y$

$\text{DIST} = 1 - y$

$$\boxed{\int_0^1 (1-y)\, 40\pi \sin^{-1}(y)\, dy}$$

(b) (4 points) Now evaluate the integral in part (a). Give your answer in exact form.

$40\pi \int_0^1 (1-y)\sin^{-1}(y)\,dy$ $\quad u = \sin^{-1}(y) \quad dv = 1-y\, dy$

$du = \frac{1}{\sqrt{1-y^2}}dy \quad v = y - \frac{1}{2}y^2$

$40\pi\left[(y - \frac{1}{2}y^2)\sin^{-1}(y)\Big|_0^1 - \int_0^1 \frac{y - \frac{1}{2}y^2}{\sqrt{1-y^2}}\,dy\right]$ $\quad y = \sin\theta$

$40\pi\left[\left((\frac{1}{2})\frac{\pi}{2} - 0\right) - \int_0^{\pi/2} \frac{\sin\theta - \frac{1}{2}\sin^2\theta}{\cancel{\cos\theta}}\cancel{\cos\theta}\,d\theta\right]$ $\quad dy = \cos\theta\, d\theta$

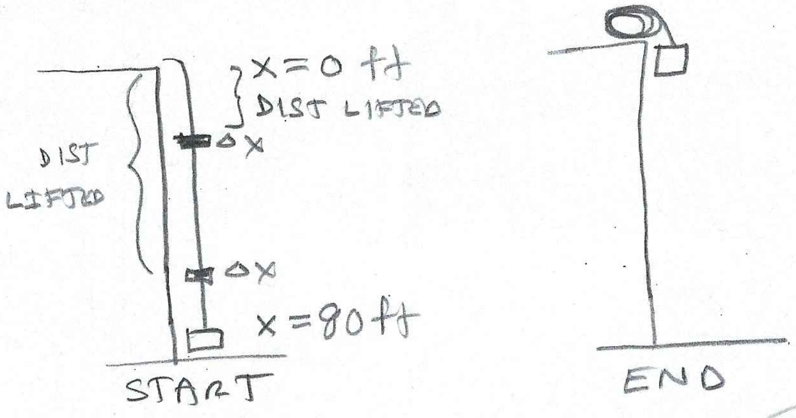
$40\pi\left[\frac{\pi}{4} + \cos\theta\Big|_0^{\pi/2} + \frac{1}{2}\int_0^{\pi/2}\sin^2\theta\, d\theta\right]$

$40\pi\left[\frac{\pi}{4} + (0-1) + \frac{1}{4}\int_0^{\pi/2} 1 - \cos(2\theta)\,d\theta\right]$

$40\pi\left[\frac{\pi}{4} - 1 + \frac{1}{4}\left(\theta - \frac{1}{2}\sin(2\theta)\Big|_0^{\pi/2}\right)\right]$

$40\pi\left[\frac{\pi}{4} - 1 + \frac{\pi}{8} - \frac{1}{8}(0)\right] = \boxed{40\pi\left[\frac{3\pi}{8} - 1\right] = 5\pi(3\pi - 8)}$

6. (10 points) An 80-ft cable is used to lift 50 pounds of coal up a mine shaft 80 ft deep. The bottom half of the cable weighs 2 pounds per foot and the top half of the cable weighs 3 pounds per foot. Find the work done in foot-pounds.



$\boxed{I}$ WORK TO LIFT COAL $= 50 \cdot 80 = \boxed{4000 \text{ ft-lbs}}$

$\boxed{II}$ CABLE FOR $0 \le x \le 40$

FORCE OF SLICE $= 3 \Delta x$

DIST. LIFTED $= x$

$$\text{WORK} = \int_0^{40} x \; 3\,dx$$
$$= \frac{3}{2} x^2 \Big|_0^{40} = \frac{3}{2}(40)^2$$
$$= \frac{3}{2} 1600 = \boxed{2400 \text{ ft-lbs}}$$

$\boxed{III}$ CABLE FOR $40 \le x \le 80$

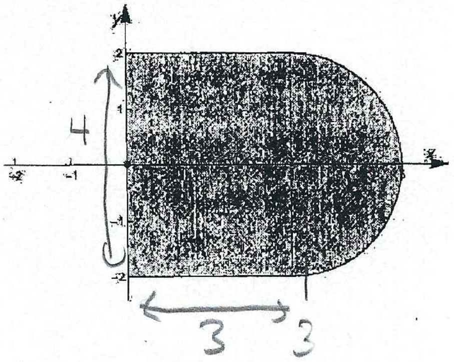
FORCE OF SLICE $= 2\Delta x$

DIST. LIFTED $= x$

$$\text{WORK} = \int_{40}^{80} x \; 2\,dx = x^2 \Big|_{40}^{80}$$
$$= (80)^2 - (40)^2$$
$$= 6400 - 1600 = \boxed{4800 \text{ ft-lbs}}$$

$$\text{TOTAL} = 4000 + 2400 + 4800$$
$$= \boxed{11{,}200 \text{ ft-lbs}}$$

3. (10 points) Consider the region in the xy-plane formed by a rectangle of height 4 and width 3 and a half-disk of radius 2 centered at $(3,0)$, as shown in the figure. Compute $\bar{x}$, the x-component of the centroid of the region.



EQUATION FOR CIRCLE
WITH CENTER (3,0) AND RADIUS 2

$$(x-3)^2+y^2=(2)^2$$

RIGHT HALF-CIRCLE

$$\Rightarrow (x-3)^2 = 4-y^2 \Rightarrow x = 3 \pm \sqrt{4-y^2}$$

- - - - - - - - - - - - - - -

IN TERMS OF x, $\bar{x} = \frac{1}{\text{AREA}} \int_a^b x\,(f(x)-g(x))dx$

★ IN TERMS OF y, $\bar{x} = \frac{1}{\text{AREA}} \int_a^b \frac{1}{2}(f(y))^2 - \frac{1}{2}(g(y))^2 dy$

- AREA $= 12 + \frac{1}{2}\pi(2)^2 = 12+2\pi$

- $\int_{-2}^{2} \frac{1}{2}\left(3+\sqrt{4-y^2}\right)^2 - \frac{1}{2}(0)^2 dy$

$$\frac{1}{2}\int_{-2}^{2} 9 + 3\sqrt{4-y^2} + 4 - y^2 dy$$

$$\frac{1}{2}\left[\int_{-2}^{2} 13-y^2 dy + 3\underbrace{\int_{-2}^{2}\sqrt{4-y^2}dy}_{\text{half-circle area}}\right]$$

$$\frac{1}{2}\left[13y - \frac{1}{3}y^3\Big|_{-2}^{2} + 3\,\frac{1}{2}\pi(2)^2\right]$$

$$\frac{1}{2}\left[\left(26-\frac{8}{3}\right)-\left(-26+\frac{8}{3}\right)+6\pi\right]$$

$$\frac{1}{2}\left[\underbrace{52-\frac{16}{3}}_{\frac{140}{3}}+6\pi\right] = \frac{70}{3}+3\pi$$

$$\boxed{\bar{X} = \frac{\frac{70}{3}+3\pi}{12+2\pi}}$$

$$= \frac{70+9\pi}{36+6\pi}$$

9. (10 points) Find the solution of the initial value problem

$$\frac{dy}{dx} = \frac{(x+3)(y+2)}{x^2+9}, \quad y(0) = 10.$$

Give your answer in the form $y = f(x)$.

$$\int \frac{1}{y+2}\,dy = \int \frac{x+3}{x^2+9}\,dx$$

$$\ln|y+2| = \int \frac{x}{x^2+9}\,dx + \int \frac{3}{x^2+9}\,dx$$

$u = x^2+9$

$du = 2x\,dx$

$$\ln|y+2| = \frac{1}{2}\int \frac{1}{u}\,du + \tan^{-1}(x/3) + C_1$$

$$\ln|y+2| = \frac{1}{2}\ln|u| + \tan^{-1}(x/3) + C_1$$

$$\ln|y+2| = \frac{1}{2}\ln(x^2+9) + \tan^{-1}(x/3) + C_1$$

$$y+2 = \pm e^{C_1}\underbrace{e^{\frac{1}{2}\ln(x^2+9)}}\,e^{\tan^{-1}(x/3)}$$

$$y = -2 + C\cdot\sqrt{x^2+9}\,e^{\tan^{-1}(x/3)}$$

$$y(0) = 10 \Rightarrow \quad 10 = -2 + C\sqrt{9}\,e^{\tan^{-1}(0)}$$

$$12 = 3C \qquad C = 4$$

$$\boxed{y = -2 + 4\sqrt{x^2+9}\,e^{\tan^{-1}(x/3)}}$$